the $b$'s being real, and $r$ being the rank of $(c_{gh})$.

$$\therefore \quad c_{gh} = \sum_{i=1}^{r} b_{ig} b_{ih}$$

$$\therefore \quad W_{gh} = \sum_{m=0}^{\infty} a_m \left( \sum_{i=1}^{r} b_{ig} b_{ih} \right) = \sum_{p_1, \dots, p_r} \underset{p_1, \dots, p_r}{C} \left(b_{1g} b_{1h}\right)^{p_1} \left(b_{2g} b_{2h}\right)^{p_2} \left(b_{rg} b_{rh}\right)^{p_r},$$

$\underset{p_1, \dots, p_r}{C}$ standing for

$$\frac{\lfloor \Sigma p}{\Pi \lfloor p} a_{\Sigma p}.$$

$\therefore$ $(W_{gh})$ is the matrix of the quadratic form

$$(1) \qquad \sum_{p_1, \dots, p_r = 0, 1, \dots} \underset{p_1, \dots, p_r}{C} \left( \sum_{j=1}^{n} b_{1j}^{p_1} \cdots b_{rj}^{p_r} y_j \right)^2,$$

which is also a positive or vanishing form, since the series in fact is absolutely convergent for arbitrary values of $y_1, \cdots, y_r$.

$\therefore |W_{gh}| \geqq 0$, which is the first part of the generalized theorem.

Suppose now that $|W_{gh}| = 0$, so that the form (1) is only semi-definite. The $C$'s being now all positive, it follows that any form obtained from (1) by giving to the $C$'s other positive values will be semi-definite. But $w_{gh}^k = \exp\,(kc_{gh})$, which differs from $W_{gh}$ only in the values of the (positive) coefficients, $k$ being positive. $\therefore$ $(w_{gh}^k)$ is the matrix of a semi-definite form. Therefore

$$(2) \qquad |w_{gh}^k| = 0, \qquad (k = 1, 2, \cdots).$$

Now every minor of $|w_{gh}|$ of the type $\begin{vmatrix} w_{gg} & w_{gh} \\ w_{gh} & w_{hh} \end{vmatrix}$ must be $\geqq 0$. If all such are $> 0$, then for every $g$ and $h$ $(g \neq h)$, $w_{gh}$ has an absolute value $< \sqrt{w_{gg} w_{hh}}$. $\therefore$ the leading term of the determinant $|w_{gh}|$ is greater in absolute value than any other term and $> 1$. $\therefore \lim_{k=\infty} |w_{gh}^k| = \infty$, which contradicts (2). $\therefore$ for some $g$ and $h$ $(g \neq h)$, $w_{gg} w_{hh} = w_{gh}^2$.

That is $c_{gg} + c_{hh} = 2c_{gh}$, or $\int_0^1 (\phi_g - \phi_h)^2 dx = 0$. Hence, $\phi_g(x) = \phi_h(x)$ on (01).

**422. Proposed by O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.**

Prove that

$$\int_0^1 \int_0^1 f(xy)(1 - x)^{m-1} y^m (1 - y)^{n-1} dx dy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} \int_0^1 f(z)(1 - z)^{m+n-1} dz,$$

$f(xy)$ being an arbitrary function of $xy$.

## Solution by the Proposer.

Let us arrange the integral as follows:

$$I = \int_0^1 dy \int_0^1 f(xy)(y - xy)^{m-1}(1 - y)^{n-1} y dx,$$

the integration with respect to $x$ to be made first, considering $y$ as a constant.

Let $z = xy$, then $dz = ydx$. This gives us

$$I = \int_0^1 dy \int_0^y f(z)(y - z)^{m-1}(1 - y)^{n-1} dz.$$

Changing the order of integrations, we have

$$I = \int_0^1 f(z) dz \int_z^1 (y - z)^{m-1}(1 - y)^{n-1} dy.$$

For a justification of this step see Bôcher's *Integral Equations*, p. 4, Dirichlet's Formula.

Now let

$$y - z = (1 - z)w, \qquad dy = (1 - z)dw,$$

and

$$1 - y = (1 - z)(1 - w).$$

By substituting these results, we obtain

$$I = \int_0^1 f(z)(1 - z)^{m+n-1}dz \int_0^1 w^{m-1}(1 - w)^{n-1}dw = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} \int_0^1 f(z)(1 - z)^{m+n-1}dz.$$

This problem is taken from Whittaker & Watson's *Modern Analysis*, p. 250. Given in *Math. Trip.*, 1894.

*Note.*—A second solution was received but no name signed to it. EDITOR.

**423. Proposed by J. B. REYNOLDS, Lehigh University.**

Show that the envelope of all circles with their centers on the circle $x^2 + y^2 = a^2$ and tangent to the $x$-axis is the two-arched epicycloid.

## SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The general form of equation of circles fulfilling the conditions of the problem is

$$f(x, y, x_1, r) = (x - x_1)^2 + (y - r)^2 - r^2 = 0, \tag{1}$$

with the conditional equation

$$\varphi(x_1, r, a) = x_1^2 + r^2 - a^2 = 0. \tag{2}$$

We are to find the envelope of the system of circles represented by (1), $x_1$, $r$ being the variable parameters. Using the undetermined multiplier $\lambda$ and the auxiliary equations

$$\frac{df}{dx_1} = \lambda \frac{d\varphi}{dx_1}, \tag{3}$$

$$\frac{df}{dr} = \lambda \frac{d\varphi}{dr}, \tag{4}$$

we have

$$-(x - x_1) = \lambda x_1, \tag{5}$$

and

$$-(y - r) - r = \lambda r. \tag{6}$$

We must eliminate $x_1$, $r$, $\lambda$ from (1), (2), (5), and (6). Substituting the values of $x - x_1$, and $y - r$ from (5) and (6) in (1),

$$\lambda^2 x_1^2 + r^2(1 + \lambda)^2 = r^2. \tag{7}$$

Also eliminating $x_1$ from (2) and (7),

$$\lambda(2r^2 + a^2\lambda) = 0. \tag{8}$$

The value of $\lambda = 0$ is irrelevant, but the second factor in (8) gives

$$r^2 = -\tfrac{1}{2}a^2\lambda. \tag{9}$$

Substituting (9) in (2) gives

$$x_1^2 = \frac{a^2}{2}(\lambda + 2) \tag{10}$$

and (5) gives

$$x_1 = \frac{x}{1 - \lambda}, \tag{11}$$

which with (10) gives the cubic in $\lambda$

$$a^2\lambda^3 - 3a^2\lambda + 2(a^2 - x^2) = 0. \tag{12}$$

Again, (6) gives

$$r = \frac{y}{\lambda}, \tag{13}$$

and this with (9) gives the second cubic in $\lambda$,

$$a^2\lambda^3 + 2y^2 = 0. \tag{14}$$